Vector Spaces and Subspaces Linear Algebra X. Du

- What exactly is a vector?
- Common vector spaces: real numbers, complex numbers, *n*th-degree polynomials, R^n
- Axioms of vector spaces
  - $\vec{u},\vec{v} \in V \Rightarrow \vec{u}+\vec{v} \in V$
  - $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ Commutativity of addition
  - $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ Associativity of addition
  - $\exists\vec{0} \in V : \forall\vec{v} \in V, \vec{v}+\vec{0}=\vec{0}+\vec{v}=\vec{v}$ Addition identity
  - $\forall\vec{v} \in V, \exists -\vec{v} \in V : \vec{v}+(-\vec{v})=(-\vec{v})+\vec{v}=\vec{0}$ Additive inverse
  - $\vec{v} \in V \Rightarrow k\vec{v} \in V$
  - $1\vec{v}=\vec{v}$ Multiplicative identity
  - $a(b\vec{v})=(ab)\vec{v}$
  - $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$ Distributivity of vector addition
  - $(a+b)\vec{v}=a\vec{v}+b\vec{v}$ Distributibity of scalar addition
- **Subspaces** are subsets of vector spaces that are also vector spaces
- Axioms of subspaces (*W* is a subspace of *V*)
  - $\vec{0} \in W$
  - $\vec{u},\vec{v} \in W \Rightarrow \vec{u}+\vec{v} \in W$
  - $\vec{v} \in W \Rightarrow k\vec{v} \in W$
- A set of vectors $S=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_n\}$ **spans** *V* iff any vector in *V* can be expressed as a linear combinations of vectors in *S*, i.e. $\forall\vec{v} \in V, \exists c_1,c_2,...,c_n : \vec{v}=c_1\vec{v}_1+c_2\vec{v}_2+...+c_n\vec{v}_n$
  - *S* is also called a **spanning set**.
  - span(*S*) is the set of all vectors spanned by *S*.
- **Row space**: the vector space spanned by the rows of a matrix
  - Subspace of R^n (given $m\times n$ matrix)
- **Column space**: the vector space spanned by the columns of a matrix
  - Subspace of R^m (given $m\times n$ matrix)
  - A system of linear equations $A\vec{x}=\vec{b}$ iff $\vec{b}$ is in the column space of *A*.
- **Null space**: the space of all $\vec{x}$ such that $A\vec{x}=\vec{0}$
  - Subspace of R^n (given $m\times n$ matrix)
- Given a matrix in row-echelon form, the rows with pivot entries form a basis for the row space, and the columns with the pivot entries form a basis for the column space.
- **Rank**: rank(*A*)
  - Number of pivots of *A* in row-echelon form
  - Dimension of row space and column space
- **Nullity**: null(*A*)
  - Number of non-pivot columns of *A* in row-echelon form
  - Dimension of the null space
- **Rank-Nullity Theorem**
  - rank(*A*) + null(*A*) = *n*, where *n* is the number of columns of *A*
  - For linear transformations: dim(range(*T*)) + dim(ker(*T*)) = dim(*V*)